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Completely realizable groups

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Given a construction f on groups, we say that a group G is *f-realisable* if there is a group H such that $G \cong f(H)$, and *completely f-realisable* if there is a group H such that $G \cong f(H)$ and every subgroup of G is isomorphic to $f(H_1)$ for some subgroup H_1 of H and vice versa. In this paper, we determine completely Aut-realisable groups. We also study f-realisable groups for $f = Z, F, M, D, \Phi$, where $Z(H)$, $F(H)$, $M(H)$, $D(H)$ and $\Phi(H)$ denote the center, the Fitting subgroup, the Chermak–Delgado subgroup, the derived subgroup and the Frattini subgroup of the group H, respectively.

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1. Introduction

In group theory, there are many constructions f which start from a group H and produce another group $f(H)$. Examples of such group-theoretical constructions are: center, central quotient, derived quotient, Frattini subgroup, Fitting subgroup, Chermak–Delgado subgroup, automorphism group, Schur multiplier, other cohomology groups and various constructions from permutation groups. For each of these constructions, there is an inverse problem:

$$\text{Given a group } G, \quad \text{is there a group } H \text{ such that } G \cong f(H)? \tag{1}$$

Several new results related to this problem have been obtained in [1, 2, 6, 9, 10] for $f(H) = D(H)$, the derived subgroup of H. Note that in these papers the group

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H with the property $G \cong D(H)$ has been called an *integral* of G by analogy with calculus. Moreover, we recall [1, Problem 10.19] that asks to classify the groups in which all subgroups are integrable. It constitutes the starting point for our discussion.

Other results of the same type are given by [7, 12, 21–23] for $f(H) = \Phi(H)$, the Frattini subgroup of H. In this case, there is a precise characterization of finite groups G for which (1) has solutions, namely

$$G \cong \Phi(H) \text{ for some group } H \text{ if and only if } \mathrm{Inn}(G) \subseteq \Phi(\mathrm{Aut}(G))$$

(see [7]).

The same problem for $f(H) = \mathrm{Aut}(H)$, the automorphism group of H, has been studied in [16, 17]. We also recall the well-known class of *capable groups*, i.e. the groups G such that (1) has solutions for $f(H) = \mathrm{Inn}(H)$, the inner automorphism group of H. Their study was initiated by Baer [3] and continued in many other papers (see e.g. [5, 8]).

Inspired by these studies, we introduce the following two notions.

Given a construction f on groups, we say that a group G is

(a) *f-realisable* if there is a group H such that $G \cong f(H)$ and
(b) *completely f-realisable* if there is a group H such that:

   (i) $G \cong f(H)$;
   (ii) $\forall\, G_1 \leq G$, $\exists\, H_1 \leq H$ such that $G_1 \cong f(H_1)$;
   (iii) $\forall\, H_1 \leq H$, $\exists\, G_1 \leq G$ such that $f(H_1) \cong G_1$.

Clearly, if f is monotone, then (i) follows from (ii) and (iii). Clearly, any subgroup of an f-realisable group is itself f-realisable. This holds, for example, for $f = D$. Also, we observe that completely D-realisable groups[a] are solutions of [1, Problem 10.19].

Throughout this paper, we assume that the above groups G and H are both finite. In Sec. 2, we will determine completely Aut-realisable groups, while in Sec. 3, we will present several results concerning completely f-realisable groups for $f = Z, F, M, D, \Phi$, where $Z(H)$, $F(H)$, $D(H)$ and $\Phi(H)$ denote the center, the Fitting subgroup, the Chermak–Delgado subgroup, the derived subgroup and the Frattini subgroup of the group H, respectively.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [13, 19, 20]. For subgroup lattice concepts we refer the reader to [18].

[a]Another suitable name for these groups would be *completely integrable groups*. For such a group G, a group H satisfying (i)–(iii) is called a *complete integral* of G.

2. Completely Aut-Realisable Groups

First of all, we recall some results of MacHale [16, 17] concerning groups H with a particular automorphism group.

Theorem 2.1. *The following hold*:

(a) *There is no group H such that $\mathrm{Aut}(H) \cong \mathbb{Z}_m$ for any odd number $m > 1$,*
(b) *There exists a group H such that $\mathrm{Aut}(H) \cong \mathbb{Z}_{p^2}$ for a prime p if and only if $p = 2$ and $H \cong \mathbb{Z}_5$ or $H \cong \mathbb{Z}_{10}$,*
(c) *There exists a group H such that $\mathrm{Aut}(H) \cong \mathbb{Z}_p^3$ for a prime p if and only if $p = 2$ and $H \cong \mathbb{Z}_{24}$,*
(d) *There exists a group H such that $\mathrm{Aut}(H) \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ for a prime p if and only if $p = 2$ and $H \cong \mathbb{Z}_{15}$ or $H \cong \mathbb{Z}_{20}$ or $H \cong \mathbb{Z}_{30}$,*
(e) *There is no group H such that $\mathrm{Aut}(H) \cong \mathbb{Z}_2^4$.*

Proof. Parts (a) and (b), respectively, follow from parts (i) and (iv)(c) of Theorem 1 in [16]. Parts (c) and (d) are given by parts (iii) and (iv) of Theorem 2 in [16]. Part (e) is a summarized version of [17, Theorem 2]. □

In [14], the solutions H of $\mathrm{Aut}(H) \cong G$ have been also determined for other important classes of groups G (see e.g. Theorem 4.2 for $G = A_n$, Theorem 4.4 for $G = S_n$ or Theorem 6.3 for $G = D_{2n}$). Also, we point out another interesting result of Ledermann and Neumann [15] which states that for every $n > 0$, there exists a bound $f(n)$ such that if G is a finite group with $|G| \geq f(n)$, then $|\mathrm{Aut}(G)| \geq n$.

We are now able to give a description of completely Aut-realisable groups.

Theorem 2.2. *A group is completely* Aut*-realisable if and only if it is an elementary abelian* 2*-group of rank at most* 3.

Proof. Let G be a completely Aut-realisable group. Then there is a group H such that:

(i) $G \cong \mathrm{Aut}(H)$;
(ii) $\forall\, G_1 \leq G$, $\exists\, H_1 \leq H$ such that $G_1 \cong \mathrm{Aut}(H_1)$;
(iii) $\forall\, H_1 \leq H$, $\exists\, G_1 \leq G$ such that $\mathrm{Aut}(H_1) \cong G_1$.

Since the groups $\mathbb{Z}_p$ with p an odd prime are not Aut-realisable by Theorem 2.1(a), it follows that $G \cong \mathrm{Aut}(H)$ is a 2-group. Then so is $\mathrm{Inn}(H) \cong \frac{H}{Z(H)}$. Let

$$|H| = p_1^{n_1} \cdots p_k^{n_k},$$

where $p_1 = 2$ and $p_2, \ldots, p_k$ are odd primes, and denote by P_i a Sylow p_i-subgroup of H, $\forall\, i = 1, \ldots, k$. Then $P_2, \ldots, P_k \subseteq Z(H)$ and therefore

$$H = A \rtimes P_1, \quad \text{where } A = \prod_{i=2}^{k} P_i.$$

On the other hand, we have $n_2 = \cdots = n_k = 1$ because $p^2 \mid |H|$ implies $p \mid |\mathrm{Aut}(H)|$ for any prime p (see e.g. [11]). Consequently,

$$A \cong \mathbb{Z}_{p_2 \cdots p_k}.$$

Assume that H is not the direct product of A and P_1. Then H contains a subgroup $H_1 \cong D_{2p_j}$ for some $j = 2, \ldots, k$. By (iii), there exists $G_1 \leq G$ such that $\mathrm{Aut}(H_1) \cong G_1$. This implies that $|\mathrm{Aut}(H_1)| = p_j(p_j - 1)$ divides $|G|$ and so p_j divides $|G|$, a contradiction. Thus we have $H = A \times P_1$.

Since A and P_1 are of coprime orders, we get

$$G \cong \mathrm{Aut}(H) \cong \mathrm{Aut}(A) \times \mathrm{Aut}(P_1) \cong \left(\prod_{i=2}^{k} \mathbb{Z}_{p_i}^{\times}\right) \times \mathrm{Aut}(P_1),$$

which shows that

$$|G| = \prod_{i=2}^{k} (p_i - 1) |\mathrm{Aut}(P_1)|.$$

Since G is a 2-group, then $p_2, \ldots, p_k$ are Fermat primes[b] and P_1 is a 2-group whose automorphism group is also a 2-group.

Let K be an abelian subgroup of P_1. If K is not cyclic, then it contains a subgroup $K_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that $\mathrm{Aut}(K_1)$ is isomorphic to a subgroup of G and therefore $|\mathrm{Aut}(K_1)| = 6$ is a power of 2, a contradiction. Thus all abelian subgroups of P_1 are cyclic, implying that P_1 is either cyclic or a generalized quaternion 2-group (see e.g. [20, (4.4)]). Since $Q_{2^{n_1}}$ possesses subgroups isomorphic to Q_8 and $|\mathrm{Aut}(Q_8)| = 24$ is not a power of 2, we infer that P_1 is cyclic, i.e. $P_1 \cong \mathbb{Z}_{2^{n_1}}$. Then

$$H \cong \mathbb{Z}_{p_2 \cdots p_k} \times \mathbb{Z}_{2^{n_1}} \cong \mathbb{Z}_{2^{n_1} p_2 \cdots p_k}$$

and so

$$G \cong \mathbb{Z}_{2^{n_1} p_2 \cdots p_k}^{\times}.$$

Note that $\mathbb{Z}_{2^{n_1}}^{\times}$ is trivial for $n_1 = 0, 1$ and $\mathbb{Z}_{2^{n_1}}^{\times} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n_1 - 2}}$ for $n_1 \geq 2$.

Assume first that $k = 1$, i.e. $H \cong \mathbb{Z}_{2^{n_1}}$ and $G \cong \mathbb{Z}_{2^{n_1}}^{\times}$. If $n_1 \geq 4$, then G contains a subgroup $G_1 \cong \mathbb{Z}_{2^2}$ and so, by Theorem 2.1(b), we know that $G_1 \cong \mathrm{Aut}(H_1)$, where $H_1 \cong \mathbb{Z}_5$ or $H_1 \cong \mathbb{Z}_{10}$. In both these cases, we have that 5 divides the order of H_1 and thus 5 divides the order of H, which is impossible because H is a 2-group. Thus $n_1 \leq 3$ and we get $G \cong \mathbb{Z}_2^r$, where $r = 0, 1, 2$.

Assume now that $k \geq 3$. Then G contains a subgroup $G_1 \cong \mathbb{Z}_2^4$. It follows that $G_1 \cong \mathrm{Aut}(H_1)$ for some subgroup $H_1 \leq H$, contradicting Theorem 2.1(e). Thus $k = 2$ and by Theorem 2.1(c) we have that 3 divides $|H|$ and so $p_2 = 3$. If $n_1 \geq 4$, then G contains a subgroup $G_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$ and Theorem 2.1(d) implies that 5 divides $|H|$, contradicting the fact that H is a 2-group. Hence $n_1 \leq 3$, leading to $G \cong \mathbb{Z}_2^r$ with $r = 1, 2, 3$.

[b]Recall that a Fermat prime is a prime number of the form $2^{2^n} + 1$, $n \in \mathbb{N}$.

Conversely, if $G \cong \mathbb{Z}_2^r$ with $r \leq 3$, then it suffices to take $H = \mathbb{Z}_2$ for $r = 0$, $H = \mathbb{Z}_4$ for $r = 1$, $H = \mathbb{Z}_8$ for $r = 2$ and $H = \mathbb{Z}_8 \times \mathbb{Z}_3$ for $r = 3$. This completes the proof. $\square$

3. Completely f-Realisable Groups, where $f = Z, F, M, D, \Phi$

The problem of determining completely f-realisable groups for $f = Z$ and $f = F$ is trivial, namely:

- A group is completely Z-realisable if and only if it is abelian.
- A group is completely F-realisable if and only if it is nilpotent.

The same thing can be also said for $f = M$. Recall that, given a finite group H, the *Chermak–Delgado measure* of a subgroup K of H is defined by

$$m_H(K) = |K||C_H(K)|.$$

Let

$$m^*(H) = \max\{m_H(K) \mid K \leq H\} \quad \text{and} \quad \mathcal{CD}(H) = \{K \leq G \mid m_H(K) = m^*(H)\}.$$

Then the set $\mathcal{CD}(H)$ forms a modular, self-dual sublattice of the subgroup lattice of H, which is called the *Chermak–Delgado lattice* of H (see [13, Theorem 1.44]). The minimal member $M(H)$ of $\mathcal{CD}(H)$ is called the *Chermak–Delgado subgroup* of H. Note that $M(H)$ is characteristic, abelian and contains $Z(H)$ by [13, Corollary 1.45]. So, a (completely) M-realisable group is abelian. Conversely, it is clear that every subgroup of an abelian group is its own Chermak–Delgado subgroup. Thus we have:

- A group is completely M-realisable if and only if it is abelian.

In what follows we will focus on the other two cases. Recall that a group G is completely D-realisable if there is a group H such that, up to isomorphism, we have

$$L(G) = \{H_1' \mid H_1 \leq H\}, \tag{2}$$

where $L(G)$ denotes the subgroup lattice of G. An important class of completely D-realisable groups is given by the following theorem.

Theorem 3.1. *All abelian groups are completely D-realisable.*

Proof. Guralnick [10] showed that if A is an abelian group of order n, then the group $H = A \wr \mathbb{Z}_2$ is an integral of A. Clearly, we have $H_1' \leq A$, for all $H_1 \leq H$. Conversely, we observe that if $A_1 \leq A$, then H contains a subgroup $H_1 \cong A_1 \wr \mathbb{Z}_2$ which is an integral of A_1. $\square$

Note that there exist completely D-realisable nonabelian groups, e.g. $G = A_4$ for which it suffices to take $H = S_4$, and even completely D-realisable nonabelian

p-groups, e.g. $G = \mathrm{He}_3$, the Heisenberg group of order 3^3, for which it suffices to take $H = \mathrm{SmallGroup}(216, 36)$.

Remark. We conjecture that A_4 is the smallest completely D-realisable nonabelian group. The dihedral groups D_{2n} with $n = 3, 4, 5, 6$ and the dicyclic group Dic_3 are not D-realisable because each of them has a characteristic cyclic subgroup which is not contained in center (see [1, Proposition 3.1]). The quaternion group Q_8 is D-realisable, a group H with $Q_8 \cong D(H)$ being necessarily a proper semidirect product $P \rtimes A$, where P is a 2-group containing a normal subgroup isomorphic to Q_8 and having $D(P)$ cyclic of order 2 or 4, and A is an abelian group of odd order[c].

Indeed, if H is a finite group with $Q_8 \cong D(H)$ and P is a Sylow 2-subgroup of H including $D(H)$, then P is normal in H and H/P is abelian. Since P and H/P are of coprime orders, the Schur–Zassenhaus theorem leads to $H \cong P \rtimes A$, where $A \cong H/P$ is abelian of odd order. Moreover, $D(P)$ is a proper subgroup of $D(H)$ because Q_8 is not p-integrable (see [2, Theorem 4.2]) and so it is cyclic of order 2 or 4.

We note that a GAP search over all groups of order less or equal that 500 gives no complete integral of Q_8.

We also note that the class of completely D-realisable groups is properly contained in the class of D-realisable groups: A_5 is D-realisable — we have $A_5 = D(S_5)$, but not completely D-realisable — it has a subgroup isomorphic to D_{10} which is not D-realisable. Since A_n have subgroups of type A_5, for all $n \geq 5$, we get the following theorem.

Theorem 3.2. *Alternating groups A_n with $n \geq 5$ are D-realisable, but not completely D-realisable.*

Next we will focus on completely Φ-realisable groups. Recall that such a group G is nilpotent and satisfies $\mathrm{Inn}(G) \subseteq \Phi(\mathrm{Aut}(G))$. Again, we have a result similar with Theorem 3.1.

Theorem 3.3. *All abelian groups are completely Φ-realisable.*

Proof. Given an abelian group G, we have to prove that there exists a group H such that:

(i) $G \cong \Phi(H)$;
(ii) $\forall\, G_1 \leq G$, $\exists\, H_1 \leq H$ such that $G_1 \cong \Phi(H_1)$;
(iii) $\forall\, H_1 \leq H$, $\exists\, G_1 \leq G$ such that $\Phi(H_1) \cong G_1$.

[c] The smallest integral of Q_8 is $\mathrm{SL}(2, 3) \cong Q_8 \rtimes \mathbb{Z}_3$ (see [2]).

Since Φ is completely multiplicative, that is

$$\Phi\left(\prod_{i=1}^{m} G_i\right) \cong \prod_{i=1}^{m} \Phi(G_i) \text{ for all groups } G_i, \quad i = 1, \ldots, m,$$

it suffices to assume that G is an abelian p-group and to prove that there exists a p-group H with the above properties. This is clear because for $G \cong \prod_{i=1}^{k} \mathbb{Z}_{p^{n_i}}$ we can choose $H \cong \prod_{i=1}^{k} \mathbb{Z}_{p^{n_i+1}}$. $\square$

Note that all Φ-realisable p-groups of order p^3 are abelian (see e.g. [12, Lemma 1]). The same thing can be also said about Φ-realisable 2-groups of order 2^4 (see e.g. [21, Theorem 1]). An example of a Φ-realisable nonabelian 5-group of order 5^4 is presented in [4, Remark 5]:

$$G = \langle x, y, z, t \mid x^5 = y^5 = z^5 = t^5 = 1, [z, t] = x, [x, t] = [y, t] = l\rangle$$

for which we can choose

$$\begin{aligned} H = \langle u, v, w, x, y, z \mid u^5 = v^5 = w^5 = x^5 = y^5 = z^5 = 1, [v, w] = [v, x] = [v, z] \\ = [x, y] = 1, [v, y] = [x, w] = [w, y] = u, [w, z] = v, [x, z] = w, [y, z] = x\rangle. \end{aligned}$$

Finally, we remark that there exist Φ-realisable groups that are not completely Φ-realisable, for example $G = \mathbb{Z}_2 \times Q_8$-we have $G = \Phi(H)$, where $H = \mathrm{SmallGroup}(64, 9)$, but G contains a subgroup isomorphic to Q_8 which is the Frattini subgroup of no group.

We end this paper by proposing the following two open problems,

Problem 1. Classify completely D-realisable and completely Φ-realisable groups.

Problem 2. Study (completely) f-realisable groups for other constructions f on groups, e.g. when $f(H)$ is the Carter subgroup of the finite solvable group H.

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